Stresses in an Incompressible Viscoelastic-Plastic Thick-Walled Cylinder

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Stresses and strains in an incompressible viscoelastic-plastic thick-walled cylinder subject to an internal pressure are determind under conditions of infinitesimal plane strain. The internal pressure is assumed to increase monotonically as a function of time approaching a finite final value. With the von Mises yield condition, which is identical with the Tresca condition under the assumption of plane strain and incompressibility, the problem is statically determined; the stresses can be determined without the knowledge of the strains, which in the present problem have to be evaluated by solving an integral equation. A numerical example is given for a material with deviatoric stress-strain relations characteristic of a standard solid with yield limit in which the integral equation is reduced to a differential equation.

1. Introduction

IN connection with stress analysis in cylindrical solid propellant exhibiting a complex inelastic behavior, an investigation[1] has been made on an elastically case-bounded thick-walled cylinder subject to internal pressure under the assumption that the cylinder material is perfectly elastic-plastic or nonlinearly elastic. It is, however, well known that the propellant material also exhibits viscoelastic properties. Hence, the present investigation deals with a thick-walled cylinder of viscoelastic material with a yield limit subject to an internal pressure.

Although a different interpretation of the von Mises yield condition is possible,[2] it is postulated in the present investigation, as in Ref. 3, that yielding takes place when stresses reach critical values whether due to elastic or to viscoelastic response.

The components of strain deviation e_{ij} can be written as the sum of the components of the viscoelastic strain e_{ij}^V and the plastic strain e_{ij}^P;

$$e_{ij} = e_{ij}^V + e_{ij}^P \tag{1}$$

where e_{ij}^V can be related[4] to the components of the deviatoric stress s_{ij} in the form

$$s_{ij} = 2G[e_{ij}^V + \int_0^t \dot{\psi}(t-\tau)e_{ij}^V(\tau)d\tau] \equiv \mathcal{L}[e_{ij}^V] \tag{2}$$

For simplicity, the third member of Eq. (2) is often employed for the second member in what follows.

The relaxation-rate-function $\dot{\psi}(t)$ (the dot indicates the time derivative) in Eq. (2) is the time derivative of the relaxation function $\psi(t)$, which is the stress response to the unit step strain input $e_{ij}(t) = H(t)$ in the form $s_{ij}(t) = 2G\psi(t)H(t)$, where $H(t)$ is the unit step function.

The advantageous use of Eq. (2) in one of moving boundary problems[5] suggests application of this form to the present problem, which also involves a moving boundary between viscoelastic and plastic regions.

A general treatment of stress analysis for linear viscoelastic material based on such integral operator stress-strain relations has been developed by Lee and Rogers.[6,7]

The particular form of the yield condition used here is the von Mises condition

$$f(\sigma_{ij}) \equiv J_2' - k^2 = 0 \tag{3}$$

where $J_2' = \frac{1}{2}s_{ij}s_{ij}$ is the second invariant of the deviatoric stress tensor (summation convention over repeated subscripts is adopted) and k is the yield stress in shear.

The flow rules associated with this yield condition are

$$\begin{aligned}\dot{e}_{ij}^P &= 0 \text{ if } f(\sigma_{ij}) < 0 \text{ or if } f(\sigma_{ij}) = 0 \text{ and } \dot{f}(\sigma_{ij}) < 0\\ \dot{e}_{ij}^P &= \lambda s_{ij} \text{ if } f(\sigma_{ij}) = 0 \text{ and } \dot{f}(\sigma_{ij}) = 0\end{aligned} \tag{4}$$

where

$$\dot{f}(\sigma_{ij}) = s_{ij}\dot{s}_{ij} \quad \text{and} \quad \lambda = s_{ij}\dot{e}_{ij}^P/2k^2 > 0 \tag{5}$$

Equations (1, 2, 4, and 5) together with the assumed incompressibility of the material introduced in the following section determine the stresses and strains.

2. General Expressions for Stresses and Strains

For simplicity, incompressibility of the cylinder material and a state of plane strain are assumed in which the von Mises and the Tresca conditions are identical, and $\epsilon_r = e_r$, $\epsilon_\theta = e_\theta$, $\epsilon_z = e_z = 0$, $\epsilon_r = \partial u/\partial r$, $\epsilon_\theta = u/r$, and

$$e_r + e_\theta = \frac{\partial u}{\partial r} + \frac{u}{r} = 0 \tag{6}$$

where ϵ_r, ϵ_θ, ϵ_z, e_r, e_θ, and e_z are, respectively, the radial, tangential, and axial components of strain and deviatoric strain, and u is the radial displacement in a cylindrical coordinate system in which r is the radius.

The inner surface of the cylinder is subject to the applied pressure

$$p(t) = p_0(1 - e^{-t/\tau_0}) \tag{7}$$

where p_0 is a constant, whereas the outer surface is free of traction.

It is assumed that p_0 is large enough to cause plastic yielding at the inner surface at $t = t^*$. As the pressure increases with time, the viscoelastic-plastic boundary r_0 moves outward and reaches the outer surface at $t = t_0$ if p_0 is not less than a critical value, whereas it never reaches the outer surface but approaches asymptotically some fixed position if p_0 is smaller than the critical value. The solution is obtained separately for time $t < t^*$ when the cylinder is entirely viscoelastic and for $t \geq t^*$ when the cylinder is partly plastic.

Equation (6) is satisfied by $u = \phi(t)/r$ where $\phi(t)$ is a function of time t only. Hence,

$$\epsilon_r = e_r = -\phi(t)/r^2 \qquad \epsilon_\theta = e_\theta = \phi(t)/r^2 \tag{8}$$

Because of Eqs. (2) and (8), the stress-strain relations are

$$\begin{aligned}s_r &= \mathcal{L}[e_r] = -(1/r^2)\mathcal{L}[\phi(t)]\\ s_\theta &= \mathcal{L}[e_\theta] = (1/r^2)\mathcal{L}[\phi(t)] \qquad s_z = 0\end{aligned} \tag{9}$$

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throughout the cylinder when $t < t^*$ and in the viscoelastic region ($r \geq r_0$) when $t \geq t^*$ since $e_{ij} = e_{ij}{}^V$ there.

The equilibrium equation

$$r\partial\sigma_r/\partial r = \sigma_\theta - \sigma_r = s_\theta - s_r \tag{10}$$

is combined with Eq. (9) and the boundary condition at the outer surface

$$\sigma_r]_{r=b} = 0 \tag{11}$$

to produce the radial stress σ_r. Then, the tangential stress σ_θ is obtained from Eq. (10) and the axial stress from $\sigma_z = (\sigma_r + \sigma_\theta)/2$:

$$\begin{gathered} \sigma_r = \left(\frac{1}{b^2} - \frac{1}{r^2}\right) \mathcal{L}[\phi(t)] \\ \sigma_\theta = \left(\frac{1}{b^2} + \frac{1}{r^2}\right) \mathcal{L}[\phi(t)] \qquad \sigma_z = \frac{\mathcal{L}[\phi(t)]}{b^2} \end{gathered} \tag{12}$$

The function $\phi(t)$ valid for $t < t^*$ is obtained from

$$-p(t) = \left(\frac{1}{b^2} - \frac{1}{a^2}\right) \mathcal{L}[\phi(t)] \tag{13}$$

where the boundary condition, Eq. (14), has been employed:

$$\sigma_r]_{r=a} = -p(t) \tag{14}$$

When $t \geq t^*$, a plastic region appears between $a \leq r \leq r_0$, in which, according to Eqs. (1) and (4),

$$\dot{e}_r{}^V = \dot{e}_r - \lambda s_r \qquad \dot{e}_\theta{}^V = \dot{e}_\theta - \lambda s_\theta \tag{15}$$

At some generic point $r = r(> a)$ at which the yield condition is reached at $t = t' > t^*$, the relations

$$e_r : e_\theta = \dot{e}_r : \dot{e}_\theta = -1:1 \tag{16}$$

are always valid because of Eq. (8).

For $t \leq t'$, $e_\theta{}^V = e_\theta$ and $e_r = e_r{}^V$ since $e_r{}^P = e_\theta{}^P = 0$. Therefore, at $t = t'$,

$$e_r{}^V : e_\theta{}^V = -1:1 \tag{17}$$

and also, from Eq. (9),

$$s_r : s_\theta = -1:1 \text{ or } s_\theta = -s_r = k \tag{18}$$

Hence, from Eqs. (15, 16, and 18), at $t = t'$,

$$\dot{e}_r{}^V : \dot{e}_\theta{}^V = -1:1 \tag{19}$$

Equations (17) and (19) result in the relation $e_r{}^V : e_\theta{}^V = -1:1$ at $t = t' + dt$, which, in turn, implies $s_r : s_\theta = -1:1$ at $t = t' + dt$ because of Eq. (2) with a linear operator $\mathcal{L}$.

This argument can be repeated to prove that Eq. (18) is always true in the plastic region. Hence, in the plastic region,

$$\sigma_r = \sigma - k \qquad \sigma_\theta = \sigma + k \qquad \sigma_z = \sigma \tag{20}$$

where σ is the mean stress $\sigma = \sigma_{kk}/3$.

Equations (20) are substituted into Eq. (10) and then the boundary condition Eq. (14) is used to obtain $\sigma = 2k \ln(r/a) - p(t) + k$. Hence, the stresses in the plastic region are

$$\begin{aligned} \sigma_r &= 2k \ln(r/a) - p(t) \\ \sigma_\theta &= 2k \ln(r/a) - p(t) + 2k \\ \sigma_z &= 2k \ln(r/a) - p(t) + k \end{aligned} \tag{21}$$

In the viscoelastic region, Eqs. (12) are valid. Because of continuity, the radial stresses in Eqs. (12) and (21) have to be identical at $r = r_0$. Hence, $\phi(t)(t \geq t^*)$ satisfies

$$2k \ln\left(\frac{r_0}{a}\right) - p(t) = \left(\frac{1}{b^2} - \frac{1}{r_0{}^2}\right) \mathcal{L}[\phi(t)] \tag{22a}$$

or

$$\mathcal{L}[\phi(t)] = \frac{2k \ln(r_0/a) - p(t)}{(1/b^2) - (1/r_0{}^2)} \tag{22b}$$

Since Eqs. (9) and (18) produce the relation

$$\mathcal{L}[\phi(t)] = 2G\left[\phi(t) + \int_0^t \dot{\psi}(t-\tau)\phi(\tau)d\tau\right] = kr_0{}^2 \tag{23}$$

$\mathcal{L}[\phi(t)]$ can be eliminated from Eqs. (22) and (23) for a transcendental equation

$$2k \ln(r_0/a) - k(r_0{}^2/b^2 - 1) - p(t) = 0 \tag{24}$$

which determines r_0 as a function of time under the specified internal pressure $p(t)$. Equation (23) is then an integral equation for $\phi(t)$ in which r_0 is now a known function of time.

For later use, $\dot{r}_0$ is derived by differentiating Eq. (24) with respect to time t:

$$\dot{r}_0 = \frac{\dot{p}(t)r_0}{2k(1 - r_0{}^2/b^2)} = \frac{p_0 r_0 e^{-t/\tau_0}}{2k\tau_0(1 - r_0{}^2/b^2)} \tag{25}$$

With the aid of Eq. (23), the stresses in the viscoelastic region Eqs. (12) can be written as

$$\begin{gathered} \sigma_r = k(1/b^2 - 1/r^2)r_0{}^2 \\ \sigma_\theta = k(1/b^2 + 1/r^2)r_0{}^2 \qquad \sigma_z = kr_0{}^2/b^2 \end{gathered} \tag{26}$$

Thus, the stresses can be obtained from Eqs. (12) and (13) when the cylinder is entirely viscoelastic and from Eqs. (21–23) when the cylinder is partly plastic. It should be noted that the stress distributions depend only on k and $p(t)$ but not on the viscoelastic properties of the material.

In fact, the comparison of the present work with the plane strain solution of incompressible ideally elastic-plastic cylinder with the von Mises (or the Tresca) yield condition[8] shows that the stress distributions in the viscoelastic-plastic cylinder are identical with those in the elastic-plastic cylinder since Eqs. (21, 26, and 24) are also valid for the latter.

The strains in the elastic-plastic cylinder, which are also of the form of Eq. (8), can be written as

$$\epsilon_\theta = -\epsilon_r = -\frac{p(t)/2Gr^2}{[(1/b^2) - (1/a^2)]} \tag{27a}$$

or

$$\phi(t) = -\frac{p(t)/2G}{[(1/b^2) - (1/a^2)]} \tag{27b}$$

when the cylinder is entirely elastic and

$$\epsilon_\theta = -\epsilon_r = \left(\frac{k}{2G}\right)\left(\frac{r_0}{r}\right)^2 \text{ or } \phi(t) = \frac{k}{2G} r_0{}^2 \tag{28}$$

with r_0 satisfying Eq. (24) when the cylinder is partly plastic.

As expected, the strains given in Eqs. (27) and (28) are not identical with those of the viscoelastic-plastic cylinder. This is easily seen since Eq. (13) is not satisfied by $\phi(t)$ in Eq. (27) and Eq. (23) is not satisfied either by $\phi(t)$ in Eq. (28).

Finally, it is necessary to show that the state of stress in the plastic region is always that of loading. Since the second part of Eq. (5) can be reduced to $\lambda = (1/k)(\dot{e}_\theta - \dot{e}_\theta{}^V)$, the following inequality must be satisfied in case of loading:

$$\dot{e}_\theta - \dot{e}_\theta{}^V > 0 \tag{29}$$

At some generic point $r(a < r \leq r_0)$, where the yield condition has been attained at $t = t'$,

$$e_\theta(r, t) + \int_0^t \dot{\psi}(t-\tau)e_\theta(r, \tau)d\tau = \frac{k}{2G}\left(\frac{r_0}{r}\right)^2$$

for $t > t^*$ because of Eqs. (8) and (23), and

$$e_\theta{}^V(r, t) + \int_0^t \dot{\psi}(t-\tau)e_\theta{}^V(r, \tau)d\tau = \frac{k}{2G}$$

for $t > t'$ because of the stress-viscoelastic strain relation in the plastic region.

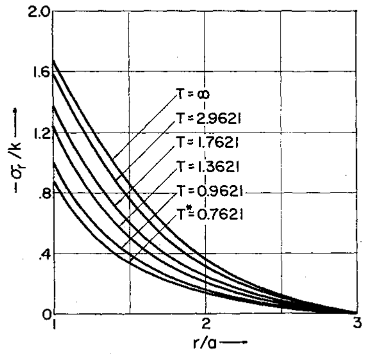

Fig. 1 Space distribution radial stress (γ_p = 0.60).

Since $e_\theta'(r, t) = e_\theta^V(r, t)$ for $t \leq t'$, it follows from the last two equations that

$$e_\theta(r, t) - e_\theta^V(r, t) + \int_{t'}^{t} \dot{\psi}(t - \tau)[e_\theta(r, \tau) - e_\theta^V(r, \tau)]d\tau = \frac{k}{2G}\left[\left(\frac{r_0}{r}\right)^2 - 1\right] \quad (30)$$

The differentiation of Eq. (30) with respect to t produces

$$\dot{e}_\theta(r, t) - \dot{e}_\theta^V(r, t) + \dot{\psi}(0)[e_\theta(r, t) - e_\theta^V(r, t)] + \int_{t'}^{t} \ddot{\psi}(t - \tau)[e_\theta(r, \tau) - e_\theta^V(r, \tau)]d\tau = \frac{k}{G}\cdot\frac{r_0\dot{r}_0}{r^2}$$

Since $r_0 \rightarrow r$ when $t \rightarrow t'$, the preceding equation reduces to

$$\dot{e}_\theta(r_0, t') - \dot{e}_\theta^V(r_0, t') = \frac{k}{G}\cdot\frac{\dot{r}_0}{r_0} \quad (31)$$

Hence, by virtue of Eq. (29), a state of loading at $r = r_0$ is assured if $\dot{r}_0 > 0$.

To prove the state of loading in the rest of the plastic region, it is sufficient to show that unloading will not occur in any part of the plastic region at any time $t(> t^*)$, under the assumption that there has been no unloading before t.

If unloading occurs in the region $r' \leq r \leq r''$, where $a \leq r' \leq r'' \leq r_0$, the inequality $\dot{s}_\theta - \dot{s}_r = \dot{\sigma}_\theta - \dot{\sigma}_r < 0$ is valid there; hence, from Eq. (10),

$$\partial\dot{\sigma}_r/\partial r < 0 \quad (32)$$

In the region where loading still continues,

$$\dot{s}_\theta - \dot{s}_r = \dot{\sigma}_\theta - \dot{\sigma}_r = 0$$

and therefore

$$\partial\dot{\sigma}_r/\partial r = 0 \quad (33)$$

Equations (32) and (33) indicate

$$\dot{\sigma}_{r_{r=r_0}} < -\dot{p}(t) \quad (34)$$

since

$$\dot{\sigma}_r = -\dot{p}(t) \quad \text{at} \quad r = a$$

On the other hand, the derivative of the first of Eqs. (26) with respect to time, evaluated at $r = r_0$ with the aid of Eq. (25), becomes

$$\dot{\sigma}_r]_{r=r} = -\dot{p}(t) \quad (35)$$

which shows a contradiction to Eq. (34). Hence, the state of stress is loading for $t > t^*$.

In summary, when $t \leq t^*$, $\phi(t)$ is obtained from Eq. (13), whereas the strains and stresses are from Eqs. (8) and (12), respectively, and when $t \geq t^*$, r_0 is obtained as a function of time from Eq. (24) or (25), and $\phi(t)$ from Eq. (23). The strains are determined from Eq. (8) and the stresses from Eq. (21) for the plastic region and from Eq. (26) for the visco-elastic region. This solution satisfies all boundary conditions and produces a state of loading for $a \leq r \leq r_0$ at any time $t(> t^*)$.

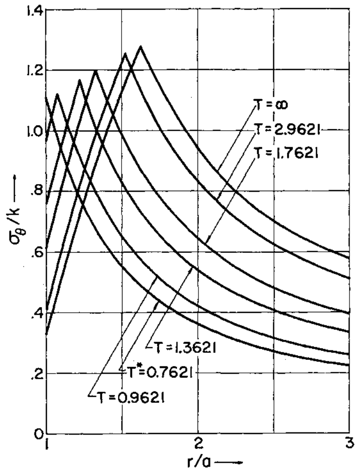

Fig. 2 Space distribution of tangential stress (γ_p = 0.60).

Since, by definition, $r_0 = b$ at $t = t_0$, t_0 is obtained from Eq. (22) as follows:

$$t_0 = -\tau_0 \ln[1 - (k/p_0)\ln\rho^2] \quad (36)$$

Therefore, the inequality

$$p_0 \geqq k \ln\rho^2 \quad (37)$$

shows the condition under which r_0 reaches b at $t_0 > t^*$, where $\rho = b/a$.

On the other hand, since $k = \mathcal{L}[\phi(t^*)]/a^2$ from Eq. (23), and $-p(t^*) = (1/b^2 - 1/a^2)\mathcal{L}[\phi(t^*)]$ from the first of Eqs. (12) at $t = t^*$,

$$t^* = -\tau_0 \ln\{1 - (k/p_0)[1 - (1/\rho^2)]\} \quad (38)$$

Therefore, the following inequality shows the values of p_0 for which $a < r_0 < b$:

$$k \ln\rho^2 > p_0 > k[1 - (1/\rho^2)] \quad (39)$$

3. Stresses and Strains for a Material with Standard Solid Stress-Strain Relation in Shear

When the stress-strain relation in shear is represented by a standard solid, the mechanical model of which is a parallel combination of an elastic spring with the modulus G_1 and a Maxwell model with the elastic modulus G_2 and the coefficient of viscosity η_2, the relaxation-rate-function $\dot{\psi}(t)$ is

$$\dot{\psi}(t) = -[(1 - \alpha)/\tau_2]e^{-t/\tau_2} \quad (40)$$

where

$$\tau_2 = \eta_2/G_2 \text{ and } \alpha = G_1/(G_1 + G_2)$$

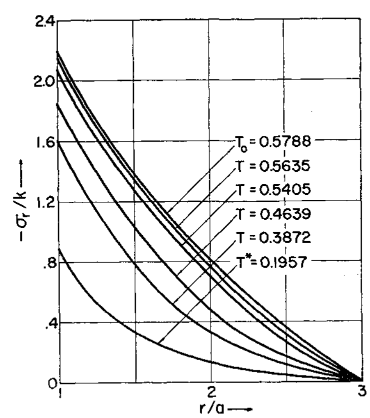

Fig. 3 Space distribution of radial stress (γ_p = 0.20).

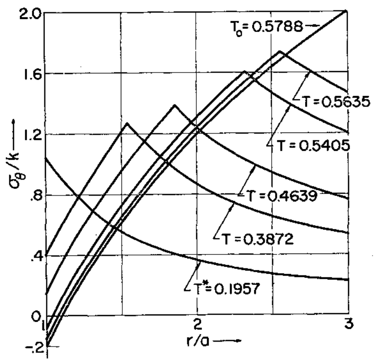

Fig. 4 Space distribution of tangential stress ($\gamma_p = 0.20$).

With Eq. (40), $\phi(t)$ for $t < t^*$ is easily obtained from Eq. (13):

$$\phi(t) = \frac{k}{2G}\frac{a^2\rho^2}{\gamma_p(\rho^2 - 1)}\left[\frac{1}{\alpha} - \beta e^{-t/\tau_0} - \left(\frac{1}{\alpha} - \beta\right)e^{-\alpha t/\tau_2}\right] \tag{41}$$

where

$$\gamma_p = k/p_0 \qquad \alpha' = \tau_2/\tau_0$$
$$\beta = (1 - \alpha')/(\alpha - \alpha') \qquad G = G_1 + G_2$$

A differential equation for $\phi(t)$ for $t > t^*$ can be derived as follows from Eq. (23):

$$\dot\phi(t) + \frac{\alpha}{\tau_2}\phi(t) = \eta(t) \equiv \frac{k}{2G}\frac{r_0^2}{\tau_2} + \frac{k}{G}r_0\dot r_0 \tag{42}$$

Hence, if $\phi(t^*)$ is the value of $\phi(t)$ in Eq. (41) at $t = t^*$,

$$\begin{aligned}\phi(t) &= e^{-\alpha t/\tau_2}\left[\int_{t^*}^{t}\eta(\tau)e^{\alpha\tau/\tau_2}\,d\tau + \phi(t^*)e^{\alpha t^*/\tau_2}\right] \\ &= a^2\gamma e^{-\alpha t/\tau_2}\left[(1-\alpha)\int_{t^*/\tau_2}^{t/\tau_2}\left(\frac{r_0}{a}\right)^2 e^{\alpha T}\,dT + \right. \\ &\left(\frac{r_0}{a}\right)^2 e^{\alpha t/\tau_2} - e^{\alpha t^*/\tau_2} + \frac{\rho^2}{\gamma_p(\rho^2-1)}\left\{\frac{1}{\alpha}e^{\alpha t^*/\tau_2} - \right. \\ &\left.\left.\beta e^{(\alpha-\alpha')t^*/\tau_2} - \left(\frac{1}{\alpha} - \beta\right)\right\}\right]\end{aligned} \tag{43}$$

where $\gamma = k/(2G)$.

For numerical computation, r_0 is evaluated as a function of time by integrating Eq. (25) step by step with the initial values $\dot p(t) = \dot p(t^*)$ and $r_0 = a$ at $t = t^*$. This is more convenient than the use of the transcendental equation (24) because of the necessity of integrating Eq. (43) numerically.

Computations are carried out on the IBM digital computer 1620 for the following set of parameter values: $\rho = b/a = 3.0$, $\alpha = G_1/G = 0.5$, $\alpha' = \tau_2/\tau_0 = 1.0$ and $\gamma_p = k/p_0 = 0.60$ or 0.20.

When $\gamma_p = 0.60$ (case I), Eq. (39) is valid, and the viscoelastic-plastic boundary does not reach the outer surface, but approach $r_0 = 1.612a$ as t increases to infinity, whereas when $\gamma_p = 0.20$ (case II), Eq. (37) is valid, and the boundary reaches the outer surface at $t = 0.5788\tau_2$. The space distributions of radial and tangential stresses are shown in Figs. 1 and 2, respectively, for case I. Similar diagrams are given for case II in Figs. 3 and 4. The nondimensional time $T = t/\tau_2$ is used for all computations with $T_0 = t_0/\tau_2$ and $T^* = t^*/\tau_2$.

The functions $\phi(T)$ represent essentially the transient behavior of strain because of its simple relation to the strain equations (8) and are plotted together with $r_0(T)$ in Fig. 5 for case I and in Fig. 6 for case II.

In these diagrams, the functions $\phi(T)$ of the elastic-plastic cylinders [Eqs. (27) and (28)] with the shear moduli equal to the unrelaxed modulus G and the relaxed modulus G_1 of the viscoelastic-plastic cylinder under consideration are plotted for comparison.

From Figs. 5 and 6, it may be concluded that if the internal pressure is built up rapidly with the viscoelastic-plastic boundary reaching the outer radius at some finite time T_0 (case II; Fig. 6), the strains are closer to those of the elastic-plastic cylinder with the unrelaxed modulus G, since

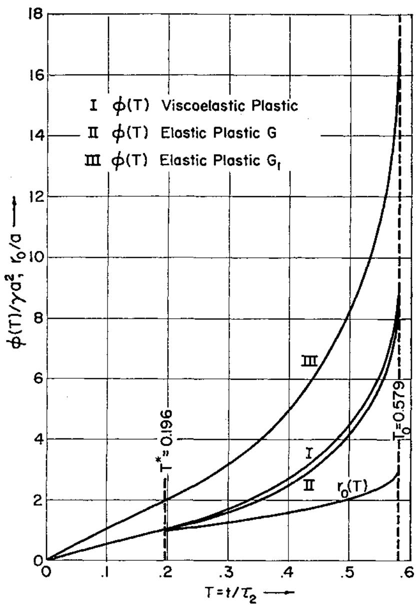

Fig. 6 $\phi(T)$ and $r_0(T)$; ($\gamma_p = 0.20$).

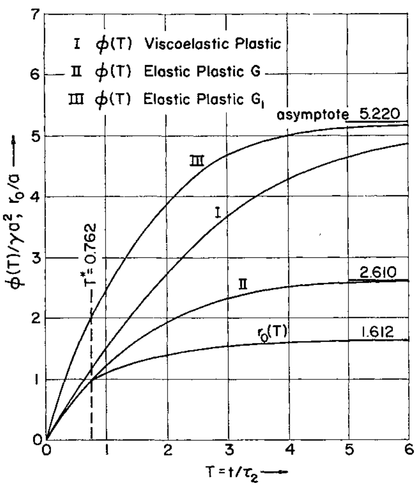

Fig. 5 $\phi(T)$ and $r_0(T)$ ($\gamma_p = 0.60$).

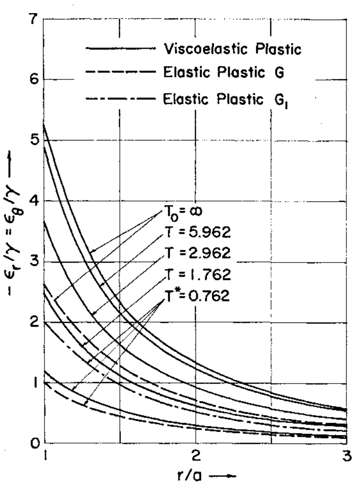

Fig. 7 Space distribution of tangential strain ($\gamma_p = 0.60$).

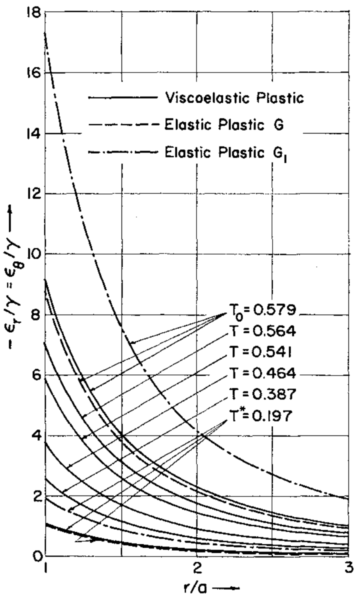

Fig. 8 Space distribution of tangential strain ($\gamma_p = 0.20$).

the time is not long enough for relaxation to take place; on the other hand, if the buildup of the internal pressure is relatively slow, with the viscoelastic-plastic boundary converging towards some position between a and b (case I; Fig. 5), the strains gradually approach those of the elastic-plastic cylinder with relaxed shear modulus G_1 and become identical as $t \to \infty$; in fact, it can be shown from Eq. (43) that [see Eq. (28)]

$$\lim_{t \to \infty} \phi(t) = \frac{k}{2G_1} r_0^2$$

Figures 7 and 8 show the space distributions of the strains. For case II, the strain distributions of the elastic-plastic cylinders are plotted at $T = T^*$ as well as at $T = T_0 = 0.5788$, whereas for case I, they are plotted at $T = T^*$ and at $T = \infty$ at which the strains of the viscoelastic-plastic and of the elastic-plastic cylinder with the relaxed shear modulus are identical.

In case II, further application of the internal pressure after $T = 0.5788$ results in the plastic flow of the entire cylinder.

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